MATHEMATICS

A REMARK ON A PAPER BY N. G. DE BRUIJN AND P. ERDÖS

BY

W. A. J. LUXEMBURG [1])

(Communicated by Prof. N. G. de Bruijn at the meeting of February 24, 1962)

1. *Introduction*

Some time ago, N. G. de Bruijn and P. Erdös in [1] obtained a theorem about coloring an infinite graph. In fact, they obtained their result from a theorem of R. Rado (see [2]) about the existence of certain choice functions.

The object of this paper is to give new proofs of this theorem about infinite graphs and the theorem of Rado. Although simple proofs of these theorems, based on Tychonoff's theorem that the Cartesian product of a family of compact sets is compact, were given earlier (see [3]), the present author believes that his method of proof may be of interest. In fact, we show that the above mentioned theorems are immediate consequences of a simple property of ultrapowers as defined in [4]. Finally, it may be of interest to remark that the use of ultrapowers eliminates the use of Zorn's lemma. Thus deducing both results directly from the ultrafilter theorem.

2. *Ultrapowers*

In this section we shall give a brief account of the theory of the very simple type of ultrapowers which are used in the following sections.

Let X and Y be two sets and let Y^X be the set of all mappings of X into Y. Let $\mathfrak{U}$ be an ultrafilter on X. For any elements $F, G \in Y^X$ we write $F \equiv_{\mathfrak{U}} G$ if and only if $\{x : x \in X \text{ and } F(x) = G(x)\} \in \mathfrak{U}$. It is easy to see that $\equiv_{\mathfrak{U}}$ is an equivalence relation on the set Y^X. The set of all equivalence classes $Y^X/\mathfrak{U}$ will be denoted by P. The reduced system P is called the *ultrapower of Y modulo* $\mathfrak{U}$. For the sake of simplicity, if $F \in Y^X$, then we shall denote the element of P to which F belongs by f.

It is easy to see that Y can be embedded in P in a one-to-one fashion by means of the equivalence classes of the constant mappings of X into Y. We shall identify Y with this subset. If Z is a subset of Y, then by Z^* we shall denote the set of all $f \in P$ such that $\{x : x \in X \text{ and } F(x) \in Z\} \in \mathfrak{U}$ for some $F \in f$. Since $\mathfrak{U}$ is a filter it follows immediately that $f \in Z^*$ if and only if for all $F \in f$ we have $\{x : x \in X \text{ and } F(x) \in Z\} \in \mathfrak{U}$. Furthermore, it is easy to see that $Z \subseteq Z^*$.

---

[1]) Work on this paper was supported by National Science Foundation Contract No. NSF–G–19914.

In the following sections the following simple result is essential.

Theorem. *If Z is a finite subset of Y, then $Z = Z^*$.*

Proof. Let $Z = \{y_1, \dots, y_n\}$, where $y_i \in Y$ $(i = 1, 2, \dots, n)$. For every $F \in Y^X$, we set $F_i = \{x : F(x) = y_i\}$. If $f \in Z^*$, then for every $F \in f$ we have that $\bigcup_{i=1}^{n} F_i \in \mathfrak{U}$. Since $\mathfrak{U}$ is an ultrafilter there exists exactly one index i $(1 \leqslant i \leqslant n)$ such that $F_i \in U$. Hence, f is the equivalence class of the constant mapping $x \to y_i$ of X into Y, i.e., $f \in Z$. This completes the proof of the theorem.

3. *A Color Problem for Infinite Graphs*

Let G be a graph and let k be a positive integer. Then G is said to be k-colorable if to each vertex one of a given set of k colors can be attached in such a way that on each edge the two-endpoints get different colors.

Theorem *(N. G. de Bruijn and P. Erdös). If G is a graph with the property that every finite subgraph is k-colorable, then G is k-colorable.*

Proof. Let X be the set of all possible k-colorings of finite subgraphs of G. Then, by hypothesis, X is not empty. For every $g \in G$, let X_g be the set of all k-colorings of finite subgraphs H of G such that $g \in H$. Then the hypothesis of the theorem implies that $X_g \neq \phi$ for all $g \in G$ and that the family $\{X_g : g \in G\}$ has the finite intersection property. Let $\mathfrak{U}$ be an ultrafilter on X which contains the family $\{X_g : g \in G\}$ and let P denote the ultrapower $\{1, 2, \dots, k\}^X/\mathfrak{U}$. We shall now define a mapping of G into P as follows: For every $g \in G$, there exists $\Phi_g \in \varphi(g)$ such that $\Phi_g(x) = x(g)\varphi$ for all $x \in X_g$. Since the set $\{1, 2, \dots, k\}$ is finite it follows from the theorem of the preceding section that for every $g \in G$ there exists a unique element $l(g) \in \{1, 2, \dots, k\}$ such that $\varphi(g) = l(g)$. We claim that the mapping $g \to l(g)$ of G into $\{1, 2, \dots, k\}$ is a k-coloring of G. Indeed, if $g_1, g_2 \in G$ are end-points of the same edge, then for all $x \in X_{g_1} \cap X_{g_2}$ we have $x(g_1) \neq x(g_2)$. Since $\mathfrak{U}$ is an ultrafilter and $X_{g_1} \cap X_{g_2} \in \mathfrak{U}$ we obtain $l(g_1) \neq l(g_2)$. This completes the proof of the theorem.

4. *A Theorem of R. Rado*

Theorem *(R. Rado). Let $\{A_\nu : \nu \in M\}$ be a non-empty family of non-empty finite sets. Assume that for every finite subset N of M a choice function a_N is given (i.e., $a_N(\nu) \in A_\nu$ for all $\nu \in N$). Then there exists a choice function a of the family $\{A_\nu : \nu \in M\}$ (i.e., a mapping a of M into $A = \bigcup(A_\nu : \nu \in M)$ such that $a(\nu) \in A_\nu$ for all $\nu \in M$) such that for every finite subset N of M there exists a finite subset N' of N with the following property: $N \subseteq N'$ and $a(\nu) = a_{N'}(\nu)$ for all $\nu \in N$.*

Proof. Let X be the set of all finite subsets of M. For every $\nu \in M$, let X_ν be the set of all $L \in X$ such that $\nu \in L$. Then the family $\{X_\nu : \nu \in M\}$

is a non-empty family of non-empty subsets of X which has the finite intersection property. Let $\mathfrak{U}$ be an ultrafilter on X which contains the family $\{X_\nu : \nu \in M\}$ and let P be the ultrapower $A^X/\mathfrak{U}$. We shall now define a mapping φ of M into P as follows: For every $\nu \in M$, there exists $\Phi_\nu \in \varphi(\nu)$ such that $\Phi_\nu(L) = a_L(\nu)$ for all $L \in X_\nu$. Since the sets A_ν are finite it follows from the theorem of section 2 that for every $\nu \in M$ there exists a unique element $a_\nu \in A_\nu$ such that $\varphi(\nu) = a_\nu$. Then the mapping a of M into A such that $a(\nu) = a_\nu$ for all $\nu \in M$ is the required choice function. Indeed, let N be a finite subset of M and let $\Phi_\nu \in \varphi(\nu)$ for every $\nu \in N$. Then the set $E_\nu = \{L : \Phi_\nu(L) = a_\nu\} \in \mathfrak{U}$ for all $\nu \in N$; and hence,

$$E = \bigcap (E_\nu : \nu \in N) \in \mathfrak{U}.$$

Finally, let $N' \in E$, then $N \subseteq N'$ and $a_{N'}(\nu) = a_\nu$ for all $\nu \in N$. This completes the proof of the theorem.

REFERENCES

1. BRUIJN, N. G. DE and P. ERDÖS, A Colour problem for infinite graphs and a problem in the theory of relations, Proc. Kon. Ned. Akad. v. Wetensch. **A, 54** = Indagationes Math., **13**, 371–373 (1951).

2. RADO, R., Axiomatic treatment of rank in infinite sets, Canad. J. Math., **1**, 337–343 (1949).

3. GOTTSCHALK, W. H., Choice functions and Tychonoff's theorem, Proc. Amer. Math. Soc., **2**, 172 (1951).

4. FRAYNE, T., D. SCOTT and A. TARSKI, Reduced products. Amer. Math. Soc. Notices, **5**, 673–674 (1958).

California Institute of Technology